

## PROJECTIVE BUNDLES OF SINGULAR PLANE CUBICS

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ABSTRACT. Classification theory and the study of projective varieties which are covered by rational curves of minimal degrees naturally leads to the study of families of singular rational curves. Since families of arbitrarily singular curves are hard to handle, it has been shown in [Keb00] that there exists a partial resolution of singularities which transforms a bundle of possibly badly singular curves into a bundle of nodal and cuspidal plane cubics.

In cases which are of interest for classification theory, the total spaces of these bundles will clearly be projective. It is, however, generally false that an arbitrary bundle of plane cubics is globally projective. For that reason the question of projectivity seems to be of interest, and the present work gives a characterization of the projective bundles.

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## 1. INTRODUCTION

Let  $C$  be a smooth algebraic curve and  $\pi : X \rightarrow C$  be a morphism from a (singular) algebraic variety to  $C$ . We assume that every fiber of  $\pi$  is isomorphic to an irreducible and reduced singular plane cubic. Omitting the word “integral” for brevity, we call this setup a “bundle of singular plane cubics”. Although  $C$  can always be covered by open subsets  $U_\alpha$  such that  $X_\alpha := \pi^{-1}(U_\alpha)$  can be identified with a family of cubic curves in  $\mathbb{P}_2 \times U_\alpha$ , it is generally not true that  $X$  can be embedded into a  $\mathbb{P}_2$ -bundle over  $C$ . In fact,  $X$  even need not be projective. The aim of this paper is to characterize those bundles which *are* projective.

This problem arises naturally in the study of projective varieties which are covered by a family of rational curves of minimal degrees: if we are given a projective variety  $V$  and a proper subvariety

$$H \subset \text{Chow}(V)$$

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parameterizing a covering family of rational curves of minimal degree, the subfamily  $H^{\text{Sing}} \subset H$  parameterizing singular rational curves is of greatest interest. It is conjectured that  $\dim H^{\text{Sing}} < \dim V - 1$ . In our previous paper [Keb00] we gave bounds on the dimension of  $H^{\text{Sing}}$  using the following line of argumentation. First, we chose a general point  $x \in V$ , considered the subfamily  $H_x^{\text{Sing}} \subset H^{\text{Sing}}$  of singular curves which contain  $x$  and constructed a diagram as follows:

$$\begin{array}{ccccc}
 \tilde{U} & \xrightarrow{\text{normalization}} & U' & \xrightarrow{\text{finite morphism}} & U \\
 & \searrow \text{\mathbb{P}_1\text{-bundle}} & \downarrow \text{bundle of singular plane cubics} & & \downarrow \pi \\
 & & \tilde{H} & \xrightarrow{\text{finite cover and normalization}} & H_x^{\text{Sing}}
 \end{array}$$

Where  $U \subset \text{Chow}(X) \times X$  is the universal family and fibers of  $\pi$  are irreducible and generically reduced singular rational curves.

Secondly, we noted that the family  $U'$  comes from the universal family over the Chow-variety and is therefore projective. We were able to show that this is possible only if the parameter space  $H_x^{\text{Sing}}$  is either finite or if it is 1-dimensional and parameterizes both nodal and cuspidal curves. The argumentation involved an analysis of the intersection of certain divisors in  $\tilde{U}$ . A complete description of projective bundles, which is somewhat more delicate and not numerical in nature, has not been given in [Keb00]. In order to complete the picture we discuss it here. It is hoped that these results will be useful in the further study of rational curves on projective varieties.

Throughout the present paper we work over the field  $\mathbb{C}$  of complex numbers and use the standard language of algebraic geometry as introduced in [Har77].

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## 2. RULED SURFACES AND ELEMENTARY TRANSFORMATIONS

The main results of this paper characterize projective bundles of singular plane cubics by describing their normalizations, which are ruled surfaces. Thus, before stating the results in section 3, it seems advisable to recall some elementary facts about the normalization morphism and about elementary transformations between ruled surfaces.

**2.1. Reduction to ruled surfaces.** As a first step in the reduction of the characterization problem, we note that to give a bundle of singular plane cubics over a smooth curve  $C$ , it is equivalent to give a ruled surface  $\tilde{X}$  over  $C$  and a double section  $\tilde{\sigma} \subset \tilde{X}$ . Indeed, if a bundle  $X$  of singular plane cubics is given, we know from [Kol96, thm. II.2.8] that its normalization will be a  $\mathbb{P}_1$ -bundle. The scheme-theoretic preimage of the (reduced) singular locus will be a double section. On the other hand, if  $\tilde{\pi} : \tilde{X} \rightarrow C$  is a  $\mathbb{P}_1$ -bundle containing a double section  $\tilde{\sigma}$ , we can

construct a diagram

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow[\text{identification}]{\gamma} & X \\
 \tilde{\pi} \searrow & & \swarrow \pi \\
 \mathbb{P}_1\text{-bundle} & & \text{bundle of singular plane cubics} \\
 & C &
 \end{array}$$

as follows. Find a cover by open subsets  $U_\alpha \subset C$  so that we can identify  $\tilde{\pi}^{-1}(U_\alpha) \cong \mathbb{P}_1 \times U_\alpha$  in a way that enables us to write

$$\tilde{\sigma} \cap \tilde{\pi}^{-1}(U_\alpha) = \{([y_0 : y_1], x) \in \mathbb{P}_1 \times U_\alpha \mid y_0^2 = g(x)y_1^2\}$$

where  $g \in \mathcal{O}(U_\alpha)$ . The identification morphism  $\gamma_\alpha = \gamma|_{\tilde{\pi}^{-1}(U_\alpha)}$  is then locally given as

$$\begin{aligned}
 \gamma_\alpha : \quad \mathbb{P}_1 \times U_\alpha &\rightarrow \mathbb{P}_2 \times U_\alpha \\
 ([y_0 : y_1], x) &\mapsto ([y_0^2 y_1 - g(x)y_1^3 : y_0^3 - g(x)y_0 y_1^2 : y_1^3], x)
 \end{aligned}$$

An elementary calculation shows that the image of  $\gamma_\alpha$  has the structure of a bundle of singular plane cubics and that the local morphisms  $\gamma_\alpha$  glue together to give a global one. More precisely, for a point  $\mu \in C$  with fiber  $X_\mu := \pi^{-1}(\mu)$ , the fiber is nodal if  $\tilde{\sigma}$  intersects  $\tilde{X}_\mu := \tilde{\pi}^{-1}(\mu)$  in two distinct points and cuspidal if it intersects in a double point. Note that the gluing morphisms do not in general come from automorphisms of  $\mathbb{P}_2$ . For that reason  $\tilde{X}$  can in general *not* be embedded into a  $\mathbb{P}_2$ -bundle over  $C$ .

**2.2. Elementary transformations.** The primary tool in the discussion of ruled surfaces will be the “elementary transformation” which is a birational map between ruled surfaces. We refer to [Har77, V.5.7.1] for the definition and a brief discussion of these maps and the associated terminology.

If  $\pi : Y \rightarrow C$  is a ruled surface and  $(\sigma_i, D_i)_{i=1\dots n}$  is a collection of sections  $\sigma_i \subset Y$  and effective divisors  $D_i \in \text{Div}(C)$  such that the supports  $|D_i|$  are mutually disjoint, we can inductively define a birational map between ruled surfaces

$$\text{elt}_{(\sigma_i, D_i)_{i=1\dots n}} : Y \dashrightarrow \tilde{Y}$$

as follows. Choose an index  $j$ , choose a closed point  $\mu \in |D_j|$  and perform an elementary transformation with center  $\pi^{-1}(\mu) \cap \sigma_j$ . Replace the  $\sigma_i$  with their strict transforms, replace  $D_i$  with  $D_i - \delta_{ij}\mu$ , where  $\delta$  is the Kronecker symbol, and start anew until all  $D_i$  are zero. It follows directly from the construction of the elementary transformation that the target variety  $\tilde{Y}$  as well as the resulting birational map are independent of the choices made.

The inverse of an elementary transformation can again be written as an elementary transformation and the following lemma shows a way to write down the inverse transformation locally. The proof is very elementary and therefore omitted here.

**Lemma 2.1.** *Let  $\pi : Y \rightarrow C$  be a ruled surface ( $C$  not necessarily compact) and let  $(\sigma_i)_{i=1\dots 3}$  be sections. Let  $D_1 \in \text{Div}(C)$  be an effective divisor and consider the birational map  $\text{elt}_{(\sigma_1, D_1)} : Y \dashrightarrow \tilde{Y}$ . If  $\tilde{\sigma}_i \subset \tilde{Y}$  are the strict transforms of the  $\sigma_i$ , then the following holds:*

1. *If  $\sigma_1, \sigma_2$  and  $\sigma_3$  are mutually disjoint, then  $D_1$  can be written as*

$$D_1 = \sum_{p \in \tilde{\sigma}_2 \cap \tilde{\sigma}_3} \text{mult}_p(\tilde{\sigma}_2, \tilde{\sigma}_3) \cdot \tilde{\pi}(p)$$

- Here  $\text{mult}_p(\tilde{\sigma}_2, \tilde{\sigma}_3)$  denotes the local intersection number of  $\tilde{\sigma}_2$  and  $\tilde{\sigma}_3$  at  $p$ .
2. If  $\sigma_1$  and  $\sigma_2$  are disjoint, then the inverse map  $\text{elt}_{(\sigma_1, D_1)}^{-1} : \tilde{Y} \dashrightarrow Y$  is given as  $\text{elt}_{(\sigma_1, D_1)}^{-1} = \text{elt}_{(\tilde{\sigma}_2, D_1)}$ .

□

A repeated application of lemma 2.1.(2) allows us to write the inverse transformation in a more complicated situation. Again we omit the proof.

**Corollary 2.2.** *Let  $\pi : Y \rightarrow C$  be a ruled surface and let  $(\sigma_i)_{i=1\dots n}$  be sections and  $D_i \in \text{Div}(C)$  be effective divisors with disjoint supports. Assume that for every index  $i$  and every point  $\mu \in |D_i|$ , there exists a unique index  $j$  such that  $\pi(D_i \cap D_j) \ni \mu$ . Consider the birational map  $\text{elt}_{(\sigma_i, D_i)_{i=1\dots n}} : Y \dashrightarrow \tilde{Y}$ . If  $\tilde{\sigma}_i \subset \tilde{Y}$  are the strict transforms of the  $\sigma_i$ , and if we set*

$$\tilde{D}_i := \sum_{j \neq i, \mu \in |D_j| \setminus \pi(\sigma_i \cap \sigma_j)} \text{mult}_\mu(D_j) \cdot \mu,$$

*then the inverse map is given as  $\text{elt}_{(\sigma_i, D_i)_{i=1\dots n}}^{-1} = \text{elt}_{(\tilde{\sigma}_i, \tilde{D}_i)_{i=1\dots n}}$ .*

□

### 3. CHARACTERIZATION OF PROJECTIVE BUNDLES

The following two theorems are the main results of this paper. Theorem 3.1 gives a construction which yields examples for projective and non-projective bundles of singular plane cubics. Theorem 3.2 shows that —after finite base change, if necessary— all projective bundles of singular plane cubics can be constructed by this method.

**Theorem 3.1.** *Let  $C$  be a smooth curve, let  $n$  a positive integer,  $(D_i)_{i=1\dots n} \in \text{Div}(C)$  arbitrary disjoint effective divisors and  $\sigma_0, (\sigma_i)_{i=1\dots n}$  and  $\sigma_\infty$  arbitrary distinct fibers of the projection  $\mathbb{P}_1 \times C \rightarrow \mathbb{P}_1$ . Construct a bundle  $X$  of singular plane cubics as follows.*

$$(3.1) \quad \begin{array}{ccccc} \mathbb{P}_1 \times C & \xrightarrow[\text{elementary transformations}]{\text{elt}_{(\sigma_i, D_i)_{i=1\dots n}}} & \tilde{X} & \xrightarrow[\text{identification}]{\gamma(\tilde{\sigma}_0, \tilde{\sigma}_\infty)} & X \\ \text{projection} \downarrow \pi_2 & & \tilde{\pi} \downarrow & & \downarrow \pi \\ C & \xlongequal{\quad} & C & \xlongequal{\quad} & C \end{array} \quad \begin{array}{l} \\ \\ \text{bundle of singular} \\ \text{plane cubics} \end{array}$$

where  $\gamma(\tilde{\sigma}_0, \tilde{\sigma}_\infty)$  is the identification morphism described in section 2.1. The bundle  $X$  is projective if and only if there exists a coordinate on  $\mathbb{P}_1$  such that  $\sigma_0 = \{[0 : 1]\} \times C$ ,  $\sigma_\infty = \{[1 : 0]\} \times C$  and  $\sigma_i = \{[\xi_i : 1]\} \times C$  where the  $\xi_i$  are roots of unity.

**Theorem 3.2.** *Let  $\pi : X' \rightarrow C'$  be a projective bundle of singular plane cubics over a smooth curve  $C'$  and assume that nodal curves occur as fibers. Then after finite base change, if necessary,  $X'$  can be constructed by the method described in theorem 3.1 above. More precisely, there exists a finite morphism  $\tau : C \rightarrow C'$  between smooth curves, there exist effective divisors  $D_i \in \text{Div}(C)$  and disjoint sections  $\sigma_0, \sigma_\infty$  and  $(\sigma_i)_{i=1\dots n} \subset \mathbb{P}_1 \times C$  such that  $X'$  can be constructed in the*

following manner.

(3.2)

$$\begin{array}{ccccccc}
 \mathbb{P}_1 \times C & \xrightarrow[\text{elementary transformations}]{\text{elt}(\sigma_i, D_i)_{i=1 \dots n}} & \tilde{X} & \xrightarrow[\text{identification}]{\gamma(\tilde{\sigma}_0, \tilde{\sigma}_\infty)} & X & \xrightarrow{\quad} & X' \\
 \text{projection} \downarrow \pi_2 & & \downarrow \tilde{\pi} & & \downarrow \pi & & \downarrow \pi' \\
 C & \xrightarrow{\quad} & C & \xrightarrow{\quad} & C & \xrightarrow[\text{finite base change}]{\tau} & C'
 \end{array}$$

bundle of singular plane cubics

Here  $X$  denotes the fibered product  $X := X' \times_{C'} C$ .

#### 4. PROJECTIVE BUNDLES

The present section is concerned with an investigation of the special geometry of projective bundles of singular plane cubics. The results will later be used in sections 5 and 6 to prove the main theorems. Throughout this section we assume that  $\pi : X \rightarrow C$  is a projective bundle of singular plane cubics over a smooth curve  $C$  and that  $L \in \text{Pic}(X)$  is an ample line bundle. Furthermore we assume that both nodal and cuspidal cubics occur as fibers of  $\pi$ . In this setup deformation theory tells us that the generic fiber has a nodal singularity and that there are only finitely many fibers with cusps.

Let  $\eta : \tilde{X} \rightarrow X$  be the normalization. By [Kol96, thm. II.2.8], the variety  $\tilde{X}$  is a  $\mathbb{P}_1$ -bundle over  $C$ .

**4.1.  $L$ -osculating points.** The restriction of the line bundle  $L$  to a fiber with nodal singularity defines a number of points which we call “ $L$ -osculating”. More precisely, we use the following definition.

**Definition 4.1.** Let  $C^N$  be a nodal plane cubic, and  $H \in \text{Pic}^k(C^N)$  be a line bundle of degree  $k > 0$ . We call a smooth point  $\sigma \in C_{\text{Reg}}^N$  an “ $H$ -osculating point” if  $\mathcal{O}(k\sigma) \cong H$ .

The following lemma shows how to calculate the  $H$ -osculating points on a given curve in a particularly simple situation.

**Lemma 4.2.** Let  $C^N$  be a nodal plane cubic and  $p \in C_{\text{Reg}}^N$  be a smooth point. Fix an identification  $\iota : \mathbb{C}^* \rightarrow C_{\text{Reg}}^N$  such that  $\iota^{-1}(p) = 1$  and set

$$(\sigma_i)_{i=1 \dots k} = \{\iota(\xi) \mid \xi^k = 1\}.$$

Then  $\sigma_i$  are the osculating points for the line bundle  $\mathcal{O}(kp)$ . In particular, there exist exactly  $k$  osculating points for  $\mathcal{O}(kp)$ .

*Proof.* Recall from [Har77, Ex. II.6.7] that the map  $\iota$  defines a group morphism

$$\begin{array}{ccc}
 \iota' : \mathbb{C}^* & \rightarrow & \text{Pic}^0(C^N) \\
 t & \mapsto & \mathcal{O}(p - \iota(t))
 \end{array}$$

It follows that  $\mathcal{O}(kp) \cong \mathcal{O}(k\iota(t))$  if and only if  $\mathcal{O}(p - \iota(t))^{\otimes k} \cong \mathcal{O}$ , i.e. if  $\iota'(t)$  is a  $k$ th root of unity.  $\square$

**4.2. The variety of  $L$ -osculating points.** In the setup of this section, the  $L$ -osculating points on the nodal fibers can be used to define a global multi-section  $\tilde{\sigma} \subset \tilde{X}$ . A detailed description of  $\tilde{\sigma}$  will be the key in our argumentation. For this, it is important to note that the relative Picard group is locally divisible.

**Proposition 4.3.** *Let  $k$  be the relative degree of the line bundle  $L$ , i.e. the intersection number of  $L$  with a fiber of  $\pi$ . Pick a point  $\mu \in C$ , fix a unit disk  $\Delta \subset C$  centered about  $\mu$  and set  $X_\Delta := \pi^{-1}(\Delta)$ . Then, after shrinking  $\Delta$ , if necessary, there exists a line bundle  $L' \in \text{Pic}(X_\Delta)$  such that  $kL' \cong L|_{X_\Delta}$ .*

*Proof.* As a first step, we will prove that  $H^2(X_\Delta, \mathbb{Z}) \cong \mathbb{Z}$ . In order to see this, recall from deformation theory that —after shrinking  $\Delta$ , if necessary—  $X_\Delta$  is of the form

$$X_\Delta \cong \{(x, [y_0 : y_1 : y_2]) \in \Delta \times \mathbb{P}_2 \mid y_2 y_1^2 - y_0^3 - f(x) y_0^2 y_2\}$$

where  $f$  is a function  $f \in \mathcal{O}(\Delta)$ . In particular, if  $N \subset X_\Delta$  is the non-normal locus and  $\tilde{N} := \eta^{-1}(N)$  its preimage in the normalization, then  $N = \Delta \times \{[0 : 0 : 1]\}$  is a unit disc and  $\tilde{N}$  either a unit disk or a union of irreducible components which are each isomorphic to  $\Delta$  and meet in a single point. In this setup, we may use the Mayer-Vietoris sequence for reduced cohomology to calculate:

$$\dots \rightarrow \underbrace{H^1(\tilde{N}, \mathbb{Z})}_{=0} \rightarrow H^2(X_\Delta, \mathbb{Z}) \rightarrow \underbrace{H^2(\tilde{X}_\Delta, \mathbb{Z})}_{=\mathbb{Z}} \oplus \underbrace{H^2(N, \mathbb{Z})}_{=0} \rightarrow \dots$$

See [BK82, prop. 3.A.7, p. 98] for more information about the sequence. Stefan Helmke pointed out that  $H^2(X_\Delta, \mathbb{Z}) \cong \mathbb{Z}$  can also be shown by deforming  $X_\Delta$  into a bundle of cuspidal plane cubics where the claim is obvious.

Now choose a section  $s \subset X_\Delta$  which is entirely supported on the smooth locus. After shrinking  $\Delta$ , this will always be possible. Consider the exponential sequence

$$(4.1) \quad \dots \longrightarrow H^1(X_\Delta, \mathcal{O}) \xrightarrow{\alpha} H^1(X_\Delta, \mathcal{O}^*) \xrightarrow{\beta} H^2(X_\Delta, \mathbb{Z}) \cong \mathbb{Z} \longrightarrow \dots$$

The element  $h := (L|_{X_\Delta} - \mathcal{O}(ks))$  satisfies  $\beta(h) = 0$  and is therefore contained in  $\text{Pic}^0(X_\Delta) = \text{Image}(\alpha)$ . Let  $h' \in \alpha^{-1}(h)$  be a preimage and note, that, since  $H^1(X_\Delta, \mathcal{O})$  is a  $\mathbb{C}$ -vector space, we can find an element  $h'' \in H^1(X_\Delta, \mathcal{O})$  such that  $h' = k \cdot h''$ . We may therefore finish by setting  $L' := \alpha(h'') \otimes \mathcal{O}(s)$ .  $\square$

The divisibility of  $L$  implies that we can locally always find a component of the osculating locus which is contained in the smooth part  $X_{\text{Reg}} \subset X$ .

**Corollary 4.4.** *Fix a point  $\mu \in C$ . If  $\Delta \subset C$  is a sufficiently small unit disk about  $\mu$ , then there exists an  $L$ -osculating section  $\sigma'_1 \subset X_\Delta$  supported in the smooth locus of  $X_\Delta$ . More precisely, there exists a section  $\sigma'_1 \subset X_{\Delta, \text{Reg}}$  such that for all points  $\mu \in \Delta$  the fiber  $X_\mu := \pi^{-1}(\mu)$  is either cuspidal or that the intersection  $\sigma'_1 \cap X_\mu$  is an  $L$ -osculating point of the fiber  $X_\mu$ .*

*Proof.* Let  $L' \in \text{Pic}(X_\Delta)$  be a line bundle such that  $kL' \cong L$ ; the existence of  $L'$  is guaranteed by proposition 4.3. Note that  $R^0\pi_*(L')$  is locally free of rank one. Thus, after shrinking  $\Delta$  if necessary, a section  $s \in H^0(X_\Delta, L')$  exists whose restriction to any fiber of  $\pi$  is not identically zero. But since the relative degree of  $L'$  is one, the restriction of  $s$  to a fiber is a section which vanishes at exactly one smooth point of the fiber. This point must therefore be  $L$ -osculating. Thus, the divisor  $\sigma_1 \in |L'|_{X_\Delta}|$

associated with the section  $s$  contains only smooth  $L$ -osculating points and maps bijectively onto the base.  $\square$

This already gives a complete description of the  $L$ -osculating locus in a neighborhood of a nodal fiber.

**Lemma 4.5.** *Let  $C^0 \subset C$  be the maximal (open) subset such that all fibers are nodal plane cubics. If  $k$  is the degree of the restriction of  $L$  to a fiber, then there exists a  $k$ -fold unbranched multisection  $\sigma' \subset X_{\text{Reg}}^0 := \pi^{-1}(C^0)$  such that the restriction to any fiber  $\sigma' \cap X_\eta$  is exactly the set of the  $L$ -osculating points of that fiber.*

*Proof.* Let  $\mu \in C^0$  be any point and  $\Delta \subset C^0$  a small unit disk centered about  $\mu$ . The preimage  $X_\Delta := \pi^{-1}(\Delta) \subset X$  will then be isomorphic to  $C^N \times \Delta$ . Let  $\sigma'_1 \subset X_{\Delta, \text{Reg}}$  be the  $L$ -osculating section whose existence is guaranteed by corollary 4.4 and find an isomorphism  $\iota : \Delta \times \mathbb{C}^* \rightarrow X_{\Delta, \text{Reg}}$  such that  $\sigma'_1 = \iota(\Delta \times \{1\})$ . Apply lemma 4.2 to see that  $\sigma'$  is then given as

$$\sigma = \{\iota(\Delta \times \{\xi_i\}) \mid \xi_i^k = 1\}.$$

Hence the claim.  $\square$

**Definition 4.6.** Let  $C^0 \subset C$  be the maximal subset such that all fibers are nodal plane cubics. Let  $\tilde{\sigma} \subset \tilde{X}$  be the closure of  $\eta^{-1}(\sigma')$ . We call the irreducible components  $(\tilde{\sigma}_i)_{i=1 \dots n} \subset \tilde{\sigma}$  the “ $L$ -osculating (multi-)sections”.

**4.3.  $L$ -osculating points in the neighborhood of a cusp.** As a next step we will find coordinates on  $\tilde{X}_\Delta := \tilde{\pi}^{-1}(\Delta)$  where the  $L$ -osculating multisection  $\tilde{\sigma}$  can be written explicitly, even if  $X_\Delta$  contains a cuspidal fiber.

**Proposition 4.7.** *Assume that the preimage  $\eta^{-1}(X_{\text{Sing}})$  consists of two distinct sections  $\tilde{\sigma}_0$  and  $\tilde{\sigma}_\infty$  and assume further that the  $L$ -osculating multisection  $\tilde{\sigma} \subset \tilde{X}$  decomposes into  $k$  irreducible components  $(\tilde{\sigma}_i)_{i=1 \dots k} \subset \tilde{X}$  (which are then sections over  $C$ ). If a point  $p \in \tilde{\sigma}_0 \cap \tilde{\sigma}_\infty$  is given, then there is a unique index  $1 \leq j \leq k$  such that  $p \notin \tilde{\sigma}_j$ . All other components  $\tilde{\sigma}_a, \tilde{\sigma}_b$  with  $a, b \notin \{0, j, \infty\}$  do contain  $p$ . If  $m = \text{mult}_p(\tilde{\sigma}_0, \tilde{\sigma}_\infty)$  is the local intersection multiplicity of  $\tilde{\sigma}_0$  and  $\tilde{\sigma}_\infty$  at  $p$ , then*

$$\text{mult}_p(\tilde{\sigma}_a, \tilde{\sigma}_b) = \text{mult}_p(\tilde{\sigma}_a, \tilde{\sigma}_0) = \text{mult}_p(\tilde{\sigma}_a, \tilde{\sigma}_\infty) = m.$$

*Proof.* Choose a unit disk  $\Delta \subset C$  centered about  $\mu := \tilde{\pi}(p)$  and equip  $\Delta$  with a coordinate  $x$ . Then  $X_\mu := \pi^{-1}(\mu)$  is a cuspidal curve. After shrinking  $\Delta$  we may assume that  $\mu$  is the only point in  $\Delta$  whose preimage is cuspidal. By corollary 4.4, we can find an index  $j$  such that  $p \notin \tilde{\sigma}_j$ . Therefore we can choose a bundle coordinate on  $\tilde{X}_\Delta \cong \Delta \times \mathbb{P}_1$  so that we can write

$$\begin{aligned} \tilde{\sigma}_0 &= \{([y_1 : y_2], x) \in \mathbb{P}_1 \times \Delta \mid y_1 = x^m y_2\} \\ \tilde{\sigma}_\infty &= \{([y_1 : y_2], x) \in \mathbb{P}_1 \times \Delta \mid y_1 = -x^m y_2\} \\ \tilde{\sigma}_j &= \{([y_1 : y_2], x) \in \mathbb{P}_1 \times \Delta \mid y_2 = 0\} \end{aligned}$$

for an integer  $m > 0$ . For a point  $\nu \in \Delta$ ,  $\nu \neq \mu$ , the map  $\eta \circ \iota_\nu$  with

$$\begin{aligned} \iota_\nu : \mathbb{C}^* &\rightarrow \tilde{\pi}^{-1}(x) \\ t &\mapsto [\nu^m(t+1) : (1-t)] \end{aligned}$$

parameterizes the smooth part of  $\pi^{-1}(\mu)$ . Apply lemma 4.2 with  $\iota := \eta \circ \iota_\nu$  and write

$$\sigma = \{([y_1 : y_2], x) \in \mathbb{P}_1 \times \Delta \mid y_1(\xi - 1) = x^m y_2(\xi + 1), \xi^k = 1\}.$$

The claim follows.  $\square$

The precise description of proposition 4.7 immediately implies that a projective bundle can be transformed into a trivial bundle in a very simple manner.

**Corollary 4.8.** *Under the assumptions of proposition 4.7 above, if we define*

$$\tilde{D}_0 := \sum_{p \in \tilde{\sigma}_0 \cap \tilde{\sigma}_\infty} \text{mult}_p(\tilde{\sigma}_0, \tilde{\sigma}_\infty) \cdot \tilde{\pi}(p)$$

*then  $\text{elt}_{(\tilde{\sigma}_0, \tilde{D}_0)} : X \dashrightarrow \hat{X}$  defines a map to the trivial bundle  $\hat{X} \cong \mathbb{P}_1 \times C$ . There exists a coordinate on  $\mathbb{P}_1$  such that the strict transforms  $\sigma_0$ ,  $\sigma_\infty$  and  $\sigma_i$  of  $\tilde{\sigma}_0$ ,  $\tilde{\sigma}_\infty$  and  $\tilde{\sigma}_i$  are of the form*

$$\begin{aligned} \sigma_0 &= \{[0, 1]\} \times C \\ \sigma_\infty &= \{[1, 0]\} \times C \\ \sigma_i &= \{[\xi_i, 1]\} \times C \end{aligned}$$

*where the  $\xi_i$  are roots of unity.*

*Proof.* It follows immediately from the construction of the elementary transformation that the intersection number between the strict transforms of  $\tilde{\sigma}_0$  and  $\tilde{\sigma}_\infty$  drops exactly by one with each transformation in the sequence  $\text{elt}_{(\tilde{\sigma}_0, \tilde{D}_0)}$ . In particular, the strict transforms  $\sigma_0, \sigma_\infty \subset \hat{X}$  of  $\tilde{\sigma}_0$  and  $\tilde{\sigma}_\infty$  are disjoint. Likewise, it follows from proposition 4.7 that the strict transforms  $\sigma_i \subset \hat{X}$  of the  $\tilde{\sigma}_i$  are sections which are mutually disjoint and disjoint from both  $\sigma_0$  and  $\sigma_\infty$ . It follows that  $\hat{X}$  must be the trivial bundle  $\hat{X} \cong \mathbb{P}_1 \times C$  and that  $\sigma_0, \sigma_\infty$  and  $\sigma_i$  are fibers of the projection  $\pi_2 : \hat{X} \rightarrow \mathbb{P}_1$ .

It remains to find the right coordinate on  $\mathbb{P}_1$ . To accomplish this, choose a general point  $\mu \in C$ . The fiber  $X_\mu := \pi^{-1}(\mu)$  will then be a nodal curve and there exists a point  $p \in X_\mu$  such that  $\mathcal{O}_{X_\mu}(kp) \cong L|_{X_\mu}$ . It follows directly from lemma 4.2 that we find coordinates on  $\tilde{X}_\mu = \eta^{-1}(X_\mu)$  such that  $\tilde{\sigma}_0 \cap \tilde{X}_\mu$  corresponds to  $[0, 1]$ ,  $\tilde{\sigma}_\infty \cap \tilde{X}_\mu$  corresponds to  $[1, 0]$ , and the  $L$ -osculating points correspond to  $[\xi_i : 1]$  where  $\xi_i$  are roots of unity. Note that  $\text{elt}_{(\tilde{\sigma}_0, \tilde{D}_0)}$  is isomorphic in a neighborhood of  $\tilde{X}_\mu$  and use the coordinates on  $\tilde{X}_\mu$  to obtain a global bundle coordinate on  $\hat{X} \cong \mathbb{P}_1 \times C \cong \tilde{X}_\mu \times C$ . This coordinate will have the desired properties, and the proof is finished.  $\square$

## 5. PROOF OF THEOREM 3.2

Corollary 4.8 already contains most arguments needed for the proof of theorem 3.2. Until the end of this section we use the notation of that theorem and fix an ample bundle  $L \in \text{Pic}(X')$ . Let  $\tau : C \rightarrow C'$  be a base change morphism which ensures that  $\eta^{-1}(X_{\text{Sing}})$  consists of two distinct sections  $\tilde{\sigma}_0$  and  $\tilde{\sigma}_\infty$ , and that the  $L$ -osculating multisection  $\tilde{\sigma} \subset \tilde{X}$  decomposes into  $k$  irreducible components  $(\tilde{\sigma}_i)_{i=1 \dots k} \subset \tilde{X}$  where  $k$  is the (positive) degree of  $L$  if restricted to a  $\pi$ -fiber. Consider the map  $\text{elt}_{(\tilde{\sigma}_0, \tilde{D}_0)} : X \dashrightarrow \hat{X} \cong \mathbb{P}_1 \times C$  which is defined in corollary 4.8 and note that we are finished if we show that the inverse transformation  $\text{elt}_{(\tilde{\sigma}_0, \tilde{D}_0)}^{-1}$  is of the form  $\text{elt}_{(\sigma_i, D_i)_{i=1 \dots n}}$  for effective divisors  $D_i \in \text{Div}(C)$  with disjoint supports



(here the  $\sigma_i$  are defined as in corollary 4.8). This, however, follows directly from corollary 2.2. Actually, the corollary shows that

$$D_i := \sum_{p \in \tilde{\sigma}_0 \cap \tilde{\sigma}_\infty, \tilde{\sigma}_i \not\ni p} \text{mult}_p(\tilde{\sigma}_0 \cap \tilde{\sigma}_\infty) \cdot \tilde{\pi}(p),$$

yield the desired transformation. This ends the proof of theorem 3.2.

## 6. PROOF OF THEOREM 3.1

**6.1. Sufficiency.** To begin the proof, we assume that  $\sigma_i$  and  $D_i$  are given as in theorem 3.1 and that there exists a coordinate on  $\mathbb{P}_1$  such that the  $\sigma_i$  correspond to roots of unity. We will then show that the variety  $X$  is projective. More precisely, we fix an index  $1 \leq i \leq n$  and let  $\tilde{\sigma}_i \subset \tilde{X}$  be the strict transform of  $\sigma_i$ . We will show that the image  $\sigma'_i := \gamma_{(\tilde{\sigma}_0, \tilde{\sigma}_\infty)}(\tilde{\sigma}_i) \subset X$  is a  $\mathbb{Q}$ -Cartier divisor. Thus, a suitable multiple of  $\sigma'_i$  generates a relatively ample line bundle, and we are done.

If the construction of theorem 3.1 involves only three sections  $\sigma_0$ ,  $\sigma_1$  and  $\sigma_\infty$ , then it is clear that the strict transform  $\tilde{\sigma}_1 \subset \tilde{X}$  of  $\sigma_1$  is disjoint from the strict transforms  $\tilde{\sigma}_0$  and  $\tilde{\sigma}_\infty$ . Thus, the image  $\sigma'_1$  does not meet the singular locus  $X_{\text{Sing}}$  of  $X$  and is therefore Cartier.

If the construction uses more than three sections and  $\sigma'_i$  is not already Cartier, let  $\mu \in C$  be a point such that  $\sigma'_i$  meets the singular locus  $X_{\text{Sing}}$  over  $\mu$ . By construction, there exists a unique index  $j$  such that  $\mu \in |D_j|$ . It follows that the strict transform  $\tilde{\sigma}_j$  of  $\sigma_j$  does not intersect  $\tilde{\sigma}_0$  or  $\tilde{\sigma}_\infty$  over  $\mu$ :

$$\tilde{\pi}(\tilde{\sigma}_0 \cap \tilde{\sigma}_j) \not\ni \mu \quad \text{and} \quad \tilde{\pi}(\tilde{\sigma}_\infty \cap \tilde{\sigma}_j) \not\ni \mu.$$

We can therefore find a suitable unit disc  $\Delta \subset C$  centered about  $p$  and we can find coordinates  $x$  on  $\Delta$  and a bundle coordinate  $[y_0 : y_1]$  such that we can write

$$\begin{aligned} \tilde{\sigma}_0 &= \{([y_0 : y_1], x) \in \mathbb{P}_1 \times \Delta \mid y_0 = x^m y_1\} \\ \tilde{\sigma}_\infty &= \{([y_0 : y_1], x) \in \mathbb{P}_1 \times \Delta \mid y_0 = -x^m y_1\} \\ \tilde{\sigma}_j &= \{([y_0 : y_1], x) \in \mathbb{P}_1 \times \Delta \mid y_1 = 0\} \end{aligned}$$

An elementary calculation, using lemma 4.2 and the assumption that there exist coordinates where  $\sigma_i$  and  $\sigma_j$  are of the form  $\{\text{Root of unity}\} \times C$  shows that

$$\begin{aligned} \tilde{\sigma}_i &= \{([y_0 : y_1], x) \in \mathbb{P}_1 \times \Delta \mid y_0 = -\frac{\xi+1}{\xi-1} x^m y_1\} \\ &= \{([y_0 : y_1], x) \in \mathbb{P}_1 \times \Delta \mid \underbrace{y_0(\xi-1) + x^m y_1(\xi+1)}_{=: f(x, y_0, y_1)} = 0\} \end{aligned}$$

where  $\xi$  is a root of unity. We fix a number  $k$  such that  $\xi^k = 1$  and we will show that  $\sigma'_i|_{X_\Delta}$  is a  $k$ -Cartier divisor, i.e.  $k \cdot \sigma'_i|_{X_\Delta}$  is Cartier. Recall from section 2.1 that the map  $\gamma$  is locally given as

$$\begin{aligned} \gamma_\Delta : \Delta \times \mathbb{C} &\rightarrow \Delta \times \mathbb{C}^2 \\ (x, y_0) &\mapsto (x, y_0^2 - x^{2m}, y_0(y_0^2 - x^{2m})) \end{aligned}$$

In particular, the image of  $\gamma_\Delta$  is isomorphic to  $\text{Spec } R$ , where  $\gamma_\Delta^\#(R) \subset k[x, y_0]$  is the subring generated by the constants  $\mathbb{C}$ , by  $x$ , by  $y_0^2$  and by the ideal  $(y_0^2 - x^{2m})$ ; see [Har77, defn. on p. 72] for the notion of  $\gamma_\Delta^\#$ . Thus, to show that  $\gamma_{(\tilde{\sigma}_0, \tilde{\sigma}_\infty)}(\tilde{\sigma}_1)$  is  $k$ -Cartier, it suffices to show that  $f(x, y_0, 1)^k \in \gamma_\Delta^\#(R)$ . We decompose  $f^k$  as

follows.

$$\begin{aligned}
f(x, y_0, 1)^k &= \sum_{i=0 \dots k} \binom{k}{i} (x^m - y_0)^i (x^m + y_0)^{k-i} \xi^{k-i} \\
&= (x^m + y_0)^k + (x^m - y_0)^k + (x^m - y_0)(x^m + y_0)(\text{rest}) \\
&= \sum_{i=0 \dots k} \binom{k}{i} [x^{m(k-i)} y_0^i + x^{m(k-i)} (-y_0)^i] + (x^m - y_0)(x^m + y_0)(\text{rest}) \\
&= 2 \cdot \underbrace{\sum_{i=0 \dots k, i \text{ even}} \binom{k}{i} x^{m(k-i)} y_0^i}_{=:A} - \underbrace{(y_0^2 - x^{2m})(\text{rest})}_{=:B}
\end{aligned}$$

It is clear each summand of  $A$  is in  $\gamma_{\Delta}^{\#}(R)$  because it involves only even powers of  $y_0$ . Likewise,  $B \in \gamma_{\Delta}^{\#}(R)$  as  $B$  is contained in the ideal  $(y_0^2 - x^{2m})$ . It follows that  $f^k \in \gamma_{\Delta}^{\#}(R)$ , and we are done.

**6.2. Necessity.** It remains to show that the conditions spelled out in theorem 3.1 are also necessary. For this assume that  $X$  is projective. We are finished if we can show that this implies the existence of a coordinate on  $\mathbb{P}_1$  such that  $\sigma_0$ ,  $\sigma_{\infty}$  and  $\sigma_i$  correspond to  $[0, 1]$ ,  $[1, 0]$  and  $[\xi_i, 1]$  for certain roots of unity  $\xi_i$ . By corollary 4.8, such a coordinate can be found if the birational map  $\text{elt}_{(\bar{\sigma}_0, \bar{D}_0)} : X \dashrightarrow \hat{X}$  which was defined in corollary 4.8 is the inverse of the map  $\text{elt}_{(\sigma_i, D_i)} : X \dashrightarrow \tilde{X}$  which was used in theorem 3.1 in order to construct the bundle  $X$ . This, however, is exactly the statement of lemma 2.1.

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